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Uniform annihilators of local cohomology

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Abstract

In this paper, we study the properties of noetherian rings containing uniform local cohomological annihilators. It turns out that all such rings should be universally catenary and locally equidimensional. We will prove a necessary and sufficient condition for such rings, which enables us to show that if a locally equidimensional ring R is the image of a Cohen–Macaulay ring, then R has a uniform local cohomological annihilator. Moreover, we will give a positive answer to a conjecture of Huneke [C. Huneke, Uniform bounds in noetherian rings, Invent. Math. 107 (1992) 203–223, Conjecture 2.13] about excellent rings with dimension no more than 5.

Keywords: Local cohomology; Cohen–Macaulay ring; Excellent rings

1. Introduction

Throughout this paper all rings are commutative, associative with identity, and noetherian. For any unexplained notation and terminology we refer the reader to [Ma].

Recall that a local noetherian ring R is equidimensional if $\dim(R) = \dim(R/P)$ for all minimal primes P of R, and thus a noetherian ring R is said to be locally equidimensional if R_m is equidimensional for every maximal ideal m of R. We will use R° to denote the complement of the union of the minimal primes of R.

Let R be a noetherian ring and I be an ideal of R. For an R-module M, we will write $\mathrm{H}^i_I(M)$ for the ith local cohomology module of M with support in $\mathrm{V}(I) = \{P \in \mathrm{Spec}(R) \mid P \supseteq I\}$. We will say an element x of R° is a uniform local cohomological annihilator of R, if for every

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maximal ideal m, x kills $\mathrm{H}^i_m(R)$ for all i less than the height of m. Moreover, we say that x is a strong uniform local cohomological annihilator of R if x is a uniform local cohomological annihilator of R_P for every prime ideal P of R.

It is well known that a nonzero local cohomology module is rarely finitely generated, even the annihilators of it are not known in general. On the other hand, it has been discovered by Hochster and Huneke that the existence of a uniform local cohomological annihilator is of great importance in solving the problems such as the existence big Cohen–Macaulay algebras [HH2] and a uniform Artin–Rees theorem [Hu]. So it is very interesting to find out more noetherian rings containing uniform local cohomological annihilators.

A traditional way of studying uniform local cohomological annihilators is to make use of the dualizing complex over a local ring. Roberts initiated this method in [Ro]. By means of this technique Hochster and Huneke [HH1] prove that if a locally equidimensional noetherian ring R is a homomorphic image of a Gorenstein ring of finite dimension, then R has a strong uniform local cohomological annihilator. It is known not every local ring has a dualizing complex, however, by passing to completion, Hochster and Huneke [HH2] show that an unmixed, equidimensional excellent local ring has a strong uniform local cohomological annihilator.

It is worth noting that a lot of results concerning the annihilators of local cohomology modules have been established in recent years [BRS,Fa1,Fa2,KS,Rag1,Rag2,Sc] and so on. Let us recall some notions before we state some of these achievements. One can refer to [BS] for details.

Given ideals I, J in a noetherian ring R and an R-module M, we set

$$\lambda_I^J(M) = \inf\bigl\{\mathrm{depth}(M_P) + \mathrm{ht}(I + P/P) \bigm| P \in \mathrm{Spec}(R) \setminus \mathrm{V}(J)\bigr\},$$
$$f_I^J(M) = \inf\bigl\{i \bigm| J^n \mathrm{H}^i_I(M) \neq 0 \text{ for all positive integers } n\bigr\}.$$

We will say that the Annihilator Theorem for local cohomology modules holds over R if $\lambda_I^J(M) = f_I^J(M)$ for every choice of the finitely generated R-module M and for every choice of ideals I, J of R. We also say that the Local–global Principle for the annihilation of local cohomology modules holds over R if $f_I^J(M) = \inf\{f_{IR_P}^{JR_P}(M_P) \mid P \in \mathrm{Spec}(R)\}$ holds for every choice of ideals I, J of R and for every choice of the finitely generated R-module M.

Faltings [Fa1] established that the Annihilator Theorem for cohomology modules holds over R if R is a homomorphic image of a regular ring or R has a dualizing complex. In [Rag1], Raghavan proved that the Local–global Principle for the annihilation of local cohomology modules holds over R if R is a homomorphic image of a regular ring. More recently, Khashyarmanesh and Salarian [KS] obtained that the Annihilator Theorem and the Local–global Principle for cohomology modules hold over a homomorphic image of a (not necessarily finite-dimensional) Gorenstein ring.

Clearly, if the Annihilator Theorem holds over a ring R, one can use it to prove the existence of some annihilator x_m of the local cohomology modules $H^i_m(R)$ for each maximal ideal m. However, the element x_m may be dependent on the choice of the maximal ideal m. Raghavan [Ra1, Theorem 3.1] established an interesting uniform annihilator theorem of local cohomology modules which states that if R is a homomorphic image of a biequidimensional regular ring of finite dimension and M a finitely generated R-module, then there exists a positive integer k (depending only on M) such that for any ideals I, J of R, we have $J^k \mathrm{H}^i_I(M) = 0$ for $i < \lambda_I^J(M)$.

Note that, in all the results mentioned above, the ring considered must be, at least, a homomorphic image of a Gorenstein ring. In this paper, we first study the properties of the rings containing uniform local cohomological annihilators. It turns out that all these rings should be universally catenary and locally equidimensional (Theorem 2.1). Due to this fact, we are able to show that a power of a uniform local cohomological annihilator is a strong uniform local cohomological annihilator (Theorem 2.2). We will establish a useful criterion for the existence of uniform local cohomological annihilators. An easy consequence of one of our main results shows that if a locally equidimensional noetherian ring R of positive dimension is a homomorphic image of a Cohen–Macaulay (abbrevation *CM*) ring of finite dimension or an excellent local ring, then R has a uniform local cohomological annihilator. This greatly generalizes a lot of known results. Especially, it gives a positive answer to a conjecture of Huneke [Hu, Conjecture 2.13] in the local case.

The technique of the paper is different from the technique used by Roberts [Ro]. The point of our technique is that a uniform local cohomological annihilator of a ring R may be chosen only dependent on the dimension of R and the multiplicity of each minimal prime ideal of R. One of our main results of the paper is the following theorem, which essentially reduces the property that a ring R has a uniform local cohomological annihilator to the same property for R/P for all minimal prime P of R. Explicitly:

Theorem 3.2. *Let R be a noetherian ring of finite dimension d. Then the following conditions are equivalent*:

(i) *R has a uniform local cohomological annihilator.*
(ii) *R is locally equidimensional, and R/P has a uniform local cohomological annihilator for each minimal prime ideal P of R.*

In Section 4, we discuss the uniform local cohomological annihilators for excellent rings. The main result of this section is the following, which shows that the conjecture of Huneke [Hu, Conjecture 2.13] is valid if the dimension of the ring considered is no more than 5.

Theorem 4.6. *Let R be a locally equidimensional excellent ring of dimension d. If $d \leqslant 5$, then R has a uniform local cohomological annihilator.*

2. Basic properties

Now we begin with studying the properties of a noetherian ring R containing a uniform local cohomological annihilator. Quite unexpectedly, it turns out that R must be locally equidimensional and universally catenary.

Let R be a noetherian ring. For a maximal ideal m of R, one can see easily from the definition of local cohomology that

$$\mathrm{H}^i_{mR_m}(R_m) = \mathrm{H}^i_m(R)$$

for $i \geqslant 0$.

Theorem 2.1. *Let R be a noetherian and $x \in R^\circ$ a uniform local cohomological annihilator of R. Then*

(i) *R is locally equidimensional.*
(ii) *R is universally catenary.*

Proof. (i) Suppose that, on the contrary, R is not locally equidimensional. It implies that there exists a maximal ideal m of R, and a minimal prime ideal P contained in m such that $\mathrm{ht}(m/P) <$

$\mathrm{ht}(m)$. Replacing R by R_m, we can assume R is a local ring and m is the unique maximal ideal of R. Let

$$0 = q \cap q_2 \cap \cdots \cap q_r$$

denote a shortest primary decomposition of the zero ideal of R, where q is P-primary. By the choice of P and m, it is clear, $r > 1$. So we can choose an element $y \notin P$ such that $yq = 0$. Consequently $y\mathrm{H}^i_m(q) = 0$ for $i \geqslant 0$.

Consider the short exact sequence

$$0 \to q \to R \to R/q \to 0.$$

It induces the following long exact sequence

$$\cdots \to \mathrm{H}^i_m(R) \to \mathrm{H}^i_m(R/q) \to \mathrm{H}^{i+1}_m(q) \to \cdots.$$

Since $x\mathrm{H}^i_m(R) = 0$ for $i < \mathrm{ht}(m)$, we conclude $xy\mathrm{H}^i_m(R/q) = 0$ for $i < \mathrm{ht}(m)$. In particular, we have

$$xy\mathrm{H}^e_m(R/q) = 0, \tag{2.1}$$

where $e = \mathrm{ht}(m/q)$. As xy is a non zero-divisor for R/q, the sequence

$$0 \to R/q \xrightarrow{xy} R/q \to R/\big(q + (xy)\big) \to 0$$

is a short exact sequence of R-modules. So we have an exact sequence

$$\mathrm{H}^e_m(R/q) \xrightarrow{xy} \mathrm{H}^e_m(R/q) \to \mathrm{H}^e_m\big(R/\big(q + (xy)\big)\big).$$

Note that $\dim(R/(q + (xy))) = e - 1$. Hence $\mathrm{H}^e_m(R/(q + (xy))) = 0$ by [Gr, Proposition 6.4]. It shows the morphism $\mathrm{H}^e_m(R/q) \xrightarrow{xy} \mathrm{H}^e_m(R/q)$ is surjective. Thus

$$\mathrm{H}^e_m(R/q) = 0$$

by (2.1), but this is impossible by [Gr, Proposition 6.4] again. Therefore, R is locally equidimensional.

(ii) To prove the conclusion, it suffices to prove R_m is universally catenary for every maximal ideal m. So we can assume that R a local ring and m is the unique maximal ideal of R. By a theorem of Ratliff (see [Ma, Theorem 15.6]), it is enough to prove R/P is universally catenary for every minimal prime ideal P.

For a fixed minimal prime ideal P, let

$$0 = q \cap q_2 \cap \cdots \cap q_r$$

denote a shortest primary decomposition of the zero ideal of R, where q is P-primary. If $r = 1$, x is a nonzero-divisor of R. Clearly, x is also a uniform local cohomological annihilator of $\hat{R}$, where $\hat{R}$ is the m-adic completions of R. Hence by (i), $\hat{R}$ is equidimensional. It follows from another theorem of Ratliff (see [Ma, Theorem 31.7]) that R is universally catenary.

If $r > 1$, choose an element $y \notin P$ as in the proof (i) such that xy is a non zero-divisor of R/q and so the image of xy in R/q is a uniform local cohomological annihilator of R/q. Just as in the case $r = 1$, we assert that R/q is universally catenary, and consequently, R/P is universally catenary. This proves (ii). □

It is easy to see that a strong uniform local cohomological annihilator of a noetherian R is also a uniform local cohomological annihilator. Conversely, we have:

Theorem 2.2. *Let R be a noetherian ring of finite dimension d and x be a uniform local cohomological annihilator of R. Then a power of x is a strong uniform local cohomological annihilator of R.*

Proof. First of all, we assume that R is a local ring with the maximal ideal m. Let $\hat{R}$ denote the completion of R with respect to m. We will prove that x is also a uniform local cohomological annihilator of $\hat{R}$. It is well known that $\mathrm{H}^i_m(R) = \mathrm{H}^i_{m\hat{R}}(\hat{R})$ for all i, and consequently $x\mathrm{H}^i_{m\hat{R}}(\hat{R}) = 0$ for $i < d$. To prove that x is a uniform local cohomological annihilator of $\hat{R}$, it suffices to prove that x is not contained in any minimal prime ideal of $\hat{R}$. Let Q be an arbitrary minimal prime ideal of $\hat{R}$. Put $P = Q \cap R$. It is clear P is a prime ideal of R. Since $\hat{R}$ is flat over R, we have

$$\mathrm{ht}(Q) = \mathrm{ht}(P) + \dim\big(\hat{R}_Q/(P\hat{R}_Q)\big)$$

by [Ma, Theorem 15.1]. Note that $\mathrm{ht}(Q) = 0$. It implies $\mathrm{ht}(P) = 0$. Since x is not contained in any minimal prime ideal of R, it follows that x does not lie in P. Consequently, $x \notin Q$. So x is not contained in any minimal prime ideal of $\hat{R}$. Therefore, we conclude that x is a uniform local cohomological annihilator of $\hat{R}$.

According to Cohen Structure Theorem for complete ring, one can write $\hat{R} = S/I$, where S is a Gorenstein local ring of dimension d. By local duality, we have $x\,\mathrm{Ext}^i_S(\hat{R}, S) = 0$ for $i > 0$. Hence

$$x\,\mathrm{Ext}^i_{S_Q}(\hat{R}_Q, S_Q) = 0$$

for every prime ideal Q of S and $i > 0$. By local duality again, we conclude that for every nonminimal prime ideal $\bar{Q}$ of $\hat{R}$,

$$x\mathrm{H}^i_{\bar{Q}}(\hat{R}_{\bar{Q}}) = 0$$

for $i < \mathrm{ht}(\bar{Q})$.

Let P be an arbitrary nonminimal prime ideal of R. Set $\mathrm{ht}(P) = r$. By Theorem 2.1, R is equidimensional, so we can choose elements $x_1, x_2, \ldots, x_r$ contained in P such that

$$\mathrm{ht}(x_1, x_2, \ldots, x_r) = r \quad \text{and} \quad \dim\big(R/(x_1, \ldots, x_r)\big) = d - r.$$

Set $I = (x_1, x_2, \ldots, x_r)$. Since $\dim(\hat{R}/I\hat{R}) = \dim(R/I)$, it is clear that $\dim(\hat{R}/I\hat{R}) = d - r$. By Theorem 2.1 again, $\hat{R}$ is equidimensional, and so $\mathrm{ht}(I\hat{R}) = r$. It follows from [HH1, Theorem 11.4(b)] that for every $i \geqslant 1$ and for all $t > 0$

$$x^{2^d-1}\mathrm{H}_i\big(x_1^t, x_2^t, \ldots, x_r^t; \hat{R}\big) = 0,$$

where $\mathrm{H}_i(x_1^t, x_2^t, \ldots, x_r^t; \hat{R})$ denotes the Koszul homology group.

Since $\hat{R}$ is faithfully flat over R, we have $x^{2^d-1}\mathrm{H}_i(x_1^t, x_2^t, \dots, x_r^t; R) = 0$ for every $i \geqslant 1$ and for all $t > 0$. Hence, we obtain $x^{2^d-1}\mathrm{H}_I^i(R) = 0$ for $i < r$. In particular,

$$x^{2^d-1}\mathrm{H}^i_{PR_P}(R_P) = x^{2^d-1}\big(\mathrm{H}_I^i(R)\big)_P = 0.$$

Therefore, x^{2^d-1} is a strong uniform local cohomological annihilator of R.

Secondly, we assume that R is not a local ring. By the above proof of the local case, we have for any maximal ideal m of R and any prime ideal $P \subseteq m$,

$$x^{2^{d_m-1}}\mathrm{H}^i_{PR_P}(R_P) = 0$$

for $i < \operatorname{ht} P$, where d_m stands for the dimension of the local ring R_m. Since $d_m \leqslant d$ for every maximal ideal m, we conclude that x^{2^d-1} is a strong uniform local cohomological annihilator of R, and this ends the proof of the theorem. $\square$

Before the end of this section, we present a necessary condition for an element to be a uniform local cohomological annihilator.

Corollary 2.3. *Let x be a uniform local cohomological annihilator of a noetherian ring R. Then R_x is a CM ring.*

Proof. By Theorem 2.2, there exists a positive integer n such that x^n is a strong uniform local cohomological annihilator of R. So we have for any prime ideal P of R, $x^n\mathrm{H}^i_{PR_P}(R_P) = 0$ for $i < \mathrm{ht}(P)$. Hence for any prime ideal P with $x \notin P$, R_P is *CM*, and consequently R_x is *CM*. $\square$

3. Equivalent conditions

In this section, we will prove one of our main result, which essentially reduces the property that a ring R has a uniform local cohomological annihilator to proving the same property for R/P for all minimal primes of R. This reducing process is very useful, it enables us to find a uniform local cohomological annihilator more easily and directly. Now, before we prove the main result of this section, we need a lemma which will play a key role in the rest of the section.

Lemma 3.1. *Let (R, m) be a noetherian local ring of dimension d, and P be a minimal prime ideal of R. Let*

$$\begin{aligned} &0 \to R/P \to R \to N_1 \to 0,\\ &0 \to R/P \to N_1 \to N_2 \to 0,\\ &\quad\vdots\\ &0 \to R/P \to N_{t-1} \to N_t \to 0 \end{aligned} \tag{3.1}$$

be a series of short exact sequences of finitely generated R-modules. Let y be an element of R such that $yN_t = 0$.

(i) *If there is an element x of R such that $x\mathrm{H}_m^i(R)=0$ for $i<d$, then $(xy)^{t^{d-1}}\mathrm{H}_m^i(R/P)=0$ for $i<d$.*
(ii) *If there is an element x of R such that $x\mathrm{H}_m^i(R/P)=0$ for $i<d$, then $x^t y\mathrm{H}_m^i(R)=0$ for $i<d$.*

Proof. Clearly, the conclusions are trivial for $d=0$, and so we assume $d>0$ in the following proof.

(i) By the choice of y, it implies

$$y\mathrm{H}_m^i(N_t)=0 \tag{3.2}$$

for all $i\geqslant 0$. We will use induction on i to prove

$$(xy)^{t^i}\mathrm{H}_m^i(R/P)=0$$

holds for $0\leqslant i<d$.

For $i=0$, it is trivial because $\mathrm{H}_m^0(R/P)=0$. Now, for $0<i<d$. Suppose that we have proved

$$(xy)^{t^{i-1}}\mathrm{H}_m^{i-1}(R/P)=0. \tag{3.3}$$

Set $k=t^{i-1}$. Let us consider the long exact sequence of local cohomology derived from the last short exact sequence in (3.1):

$$\cdots\to\mathrm{H}_m^{i-1}(R/P)\to\mathrm{H}_m^{i-1}(N_{t-1})\to\mathrm{H}_m^{i-1}(N_t)\to\cdots.$$

By (3.2) and (3.3), it follows $(xy)^k y\mathrm{H}_m^{i-1}(N_{t-1})=0$. Continue the process, one can prove $(xy)^{jk}y\mathrm{H}_m^{i-1}(N_{t-j})=0$ for $j=1,2,\ldots,t-1$. Hence by the long exact sequence of local cohomology derived from the first short exact sequence in (3.1):

$$\cdots\to\mathrm{H}_m^{i-1}(N_1)\to\mathrm{H}_m^i(R/P)\to\mathrm{H}_m^i(R)\to\cdots$$

and the condition $x\mathrm{H}_m^i(R)=0$, we have $(xy)^{(t-1)k+1}\mathrm{H}_m^i(R/P)=0$. It easy to check, $t^i\geqslant(t-1)k+1$, so it follows that $(xy)^{t^i}\mathrm{H}_m^i(R/P)=0$. This completes the inductive proof. In particular, we have proved $(xy)^{t^{d-1}}\mathrm{H}_m^i(R/P)=0$ for $i<d$.

(ii) By the condition, we have

$$x\mathrm{H}_m^i(R/P)=0 \tag{3.4}$$

for $i<d$. Set $R=N_0$.

Now, we will use induction on j to prove that

$$x^j y\mathrm{H}_m^i(N_{t-j})=0 \quad \text{for } i<d$$

hold for $0\leqslant j\leqslant t$.

For $j=0$, it is trivial by (3.2). For $j>0$, suppose that we have proved

$$x^{j-1}y\mathrm{H}_m^i(N_{t-(j-1)})=0 \tag{3.5}$$

for $i<d$.

Consider the $(t-(j-1))$th short exact sequence in (3.1), it induces the following long exact sequence

$$\cdots \to \mathrm{H}^i_m(R/P) \to \mathrm{H}^i_m(N_{t-j}) \to \mathrm{H}^i_m(N_{t-(j-1)}) \to \cdots.$$

By (3.4) and (3.5), we conclude

$$x^j y \mathrm{H}^i_m(N_{t-j}) = 0$$

for $i < d$. This completes the inductive proof. In particular, we have $x^t y \mathrm{H}^i_m(R) = 0$ for $i < d$. □

We are now ready to prove our main result of the section.

Theorem 3.2. *Let R be a noetherian ring of finite dimension d. Then the following conditions are equivalent*:

(i) *R has a uniform local cohomological annihilator.*
(ii) *R is locally equidimensional, and R/P has a uniform local cohomological annihilator for each minimal prime ideal P of R.*

Proof. (i) $\Rightarrow$ (ii). The first conclusion of (ii) comes from Theorem 2.1. Let P be an arbitrary minimal prime ideal of R. Let x be a uniform local cohomological annihilator of R. Put $t = l(R_P)$. It is easy to see that there exist finitely generated R-modules $N_1, N_2, \ldots, N_t$ such that

(1) $N_1, N_2, \ldots, N_t$ fit into a series of the following short exact sequences

$$\begin{aligned} &0 \to R/P \to R \to N_1 \to 0, \\ &0 \to R/P \to N_1 \to N_2 \to 0, \\ &\quad\vdots \\ &0 \to R/P \to N_{t-1} \to N_t \to 0. \end{aligned} \tag{3.6}$$

(2) There exists an element $y \in R \setminus P$, $yN_t = 0$.

Clearly, $(xy)^{t^{d-1}}$ is not contained in P. We will show that the image of $(xy)^{t^{d-1}}$ in R/P is a uniform local cohomological annihilator of R/P. Since R is locally equidimensional, we have $\mathrm{ht}(m/P) = \mathrm{ht}(m)$ for every maximal ideal m with $m \supseteq P$. Thus it suffices to prove that, for every maximal ideal m with $m \supseteq P$

$$(xy)^{t^{d-1}} \mathrm{H}^i_{mR_m}\big((R/P)_m\big) = 0$$

for $i < \mathrm{ht}(m)$.

Localizing the short exact sequences in (3.6) at m, we obtain the following the short exact sequences

$$\begin{aligned} &0 \to (R/P)_m \to R_m \to (N_1)_m \to 0, \\ &0 \to (R/P)_m \to (N_1)_m \to (N_2)_m \to 0, \end{aligned}$$

$$\vdots$$

$$0 \to (R/P)_m \to (N_{t-1})_m \to (N_t)_m \to 0.$$

By the choice of x, we have $x\mathrm{H}^i_{mR_m}(R_m) = 0$ for $i < \mathrm{ht}(m)$. Clearly, $y(N_t)_m = 0$. Hence by Lemma 3.1(i), we conclude $(xy)^{t^{e-1}}\mathrm{H}^i_{mR_m}((R/P)_m) = 0$ for $i < e$, where $e = \mathrm{ht}(m)$. Therefore, for every maximal ideal m with $m \supseteq P$, we have

$$(xy)^{t^{d-1}}\mathrm{H}^i_{mR_m}\big((R/P)_m\big) = 0$$

for $i < \mathrm{ht}(m)$. Hence the image of $(xy)^{t^{d-1}}$ in R/P is a uniform local cohomological annihilator of R/P, and this proves (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let $P_1, P_2, \ldots, P_r$ be all the distinct minimal prime ideals of R. For each j, $1 \leqslant j \leqslant r$, put $l(R_{P_j}) = t_j$. For a fixed j, it is easy to see that there exist finitely generated R-modules $N_1^{(j)}, N_2^{(j)}, \ldots, N_{t_j}^{(j)}$ satisfying the following two properties:

(1) $N_1^{(j)}, N_2^{(j)}, \ldots, N_{t_j}^{(j)}$ fit into a series of the following short exact sequences

$$\begin{aligned}
&0 \to R/P_j \to R \to N_1^{(j)} \to 0,\\
&0 \to R/P_j \to N_1^{(j)} \to N_2^{(j)} \to 0,\\
&\quad\vdots\\
&0 \to R/P_j \to N_{t_j-1}^{(j)} \to N_{t_j}^{(j)} \to 0.
\end{aligned}\tag{3.7}$$

(2) There exists an element $y_j \in R \setminus P$ such that $y_j N_{t_j} = 0$ and y_j lies in all P_k except P_j.

To prove the conclusion, it is enough to find an element $x \in R^\circ$, such that for every maximal ideal m

$$x\mathrm{H}^i_{mR_m}(R_m) = 0$$

for $i < \mathrm{ht}(m)$.

By the condition, for each j, there exists an element $x_j \notin P_j$ such that its image in R/P_j is a uniform local cohomological annihilator of R/P_j. Set $x = \sum x_j^{t_j} y_j$. It is easy to check x lies in no minimal prime ideal of R. We will prove that x is a uniform local cohomological annihilator of R.

Let m be an arbitrary nonminimal prime ideal of R. Put $e = \mathrm{ht}(m)$. For a fixed j, localizing the short exact sequences (3.7) at m, we obtain the following short exact sequences

$$\begin{aligned}
&0 \to (R/P_j)_m \to R_m \to \big(N_1^{(j)}\big)_m \to 0,\\
&0 \to (R/P_j)_m \to \big(N_1^{(j)}\big)_m \to \big(N_2^{(j)}\big)_m \to 0,\\
&\quad\vdots\\
&0 \to (R/P_j)_m \to \big(N_{t_j-1}^{(j)}\big)_m \to \big(N_{t_j}^{(j)}\big)_m \to 0
\end{aligned}$$

and it is clear $y_j(N_{t_j}^{(j)})_m = 0$.

If m contains P_j, we have $\mathrm{ht}(m/P_j)=e$ by the assumption that R is locally equidimensional. Thus by the choice of x_j,

$$x_j \mathrm{H}^i_{mR_m}\big((R/P_j)_m\big)=0 \tag{3.8}$$

for $i<e$. If m does not contain P_j, the statement (3.8) holds trivially. Hence by Lemma 2.2(ii), we conclude that

$$x^{t_j} y_j \mathrm{H}^i_{mR_m}(R_m)=0 \tag{3.9}$$

for $i<e$. Therefore, by the choice of x, we have

$$x\mathrm{H}^i_{mR_m}(R_m)=0 \quad \text{for } i<\mathrm{ht}(m)$$

holds for every maximal ideal m of R. So x is a uniform local cohomological annihilator of R. This proves (ii) $\Rightarrow$ (i). □

Now, we end this section by an interesting corollary, which is not known in such an extent even in the local case.

Corollary 3.3. *Let R be a locally equidimensional noetherian ring of finite positive dimension. Then R has a uniform local cohomological annihilator in any one of the following cases*:

(i) *R is the homomorphic image of a CM ring of finite dimension.*
(ii) *R is an excellent local ring.*

Proof. (i) Represent R as $R=S/I$, where S is a *CM* ring of finite dimension and I an ideal of S. By Theorem 3.2, it suffices to prove that for any minimal prime ideal P of R, R/P has a uniform local cohomological annihilator. Let Q be the prime ideal of S such that $P=Q/I$. It suffices to prove that S/Q has a uniform local cohomological annihilator.

Set $n=\mathrm{ht}(Q)$. If $n=0$, then Q is a minimal prime ideal of S. Note that 1 is a uniform local cohomological annihilator of S, the conclusion follows immediately from Theorem 3.2.

Assume that $n>0$. Since S is *CM*, we can choose a regular sequence $x_1,x_2,\ldots,x_n$ contained in Q. Thus $S/(x_1,x_2,\ldots,x_n)$ is still a *CM* ring. It is obvious that $Q/(x_1,x_2,\ldots,x_n)$ is a minimal prime ideal of $S/(x_1,x_2,\ldots,x_n)$. Thus from the case $n=0$, we assert S/Q has a uniform local cohomological annihilator by Theorem 3.2.

(ii) By [HH2, Lemma 3.2], every excellent local domain has a strong uniform local cohomological annihilator, and thus R has a strong uniform local cohomological annihilator by Theorem 3.2. □

4. Uniform local cohomological annihilators of excellent rings

In this section, we restrict our discussion to the uniform local cohomological annihilators of excellent rings. By two theorems of Hochster and Huneke [HH1, Theorems 11.3, 11.4], it is easy to see that Huneke's conjecture [Hu, Conjecture 2.13] is valid if every locally equidimensional excellent noetherian ring R of finite dimension has a strong uniform local cohomological annihilator. Since for each positive integer n, the Koszul complex of every sequence $x_1^n,x_2^n,\ldots,x_k^n$ in

R with $\mathrm{ht}(x_1, x_2, \ldots, x_k) = k$ is a complex satisfying the standard conditions on height and rank in the sense of [Hu, (2.11)], it is easy to see that the converse of this result is also true. Due to Theorems 2.2 and 3.2, Huneke's conjecture is equivalent to the following:

Conjecture 4.1. *Let R be an excellent noetherian domain of finite dimension. Then R has a uniform local cohomological annihilator.*

The conjecture is known to be true if R is an excellent normal domain of dimension $d \leqslant 3$ [Hu, Proposition 4.5(vii)]. We will prove that Conjecture 4.1 is true for the ring R with $\dim(R) \leqslant 5$. In order to prove the main result of this section, we need the following result established by Goto [Go, Theorem 1.1].

Proposition 4.2. *Let (R, m) be a local ring of dimension d and x is an element of m with $(0 : x) = (0 : x^2)$. Then the following conditions are equivalent.*

(i) *$R/x^n R$ is a CM ring of dimension $d - 1$ for every integer $n > 0$.*
(ii) *$R/x^2 R$ is a CM ring of dimension $d - 1$.*

Another important result we need is the following explicit version of Corollary 3.3(ii), which follows from [HH2, Lemma 3.2].

Proposition 4.3. *Let (R, m) be an excellent local domain of dimension $d > 0$ and x be a nonzero element of R such that R_x is a CM ring. Then a power of x is a uniform local cohomological annihilator of R.*

Proof. Let $x_1, x_2, \ldots, x_d$ be an arbitrary system of parameters in m. By [HH2, Lemma 3.2], there exists a positive integer n such that x^n kills all the higher Koszul homology $\mathrm{H}_i(x_1^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d}, R)$, $i > 0$, for all positive integers $n_1, n_2, \ldots, n_d$. Hence

$$x^n \mathrm{H}_m^i(R) = \lim_{t \to \infty} \mathrm{H}_{d-i}\big(x_1^t, x_2^t, \ldots, x_d^t, R\big) = 0$$

for $i < d$. It shows that x^n is a uniform local cohomological annihilator of R. □

Let R be an excellent ring of dimension $d > 0$. For any prime ideal P of R, the regular locus of R/P is a nonempty open subset of $\mathrm{Spec}(R/P)$, and so there exists a nonempty open subset U of $\mathrm{Spec}(R/P)$ such that for any $Q \in U$, $(R/P)_Q$ is *CM*. By Nagata Criterion for openness, we conclude that the *CM* locus of R is open in $\mathrm{Spec}(R)$ (see [Ma, Theorem 24.5]). Moreover, as every minimal prime ideal P of R lies in the *CM* locus of R, we assert that the *CM* locus of R is a nontrivial open set in $\mathrm{Spec}(R)$. Hence we can choose an element $x \in R^\circ$ such that R_x is a *CM* ring. By Proposition 4.3, for any maximal ideal m of R, there exists a positive integer n_m such that $x^{n_m} \mathrm{H}_m^i(R) = 0$ for $i < \mathrm{ht}(m)$. Clearly, the positive integer n_m may be dependent on m. To solve Conjecture 4.1, it suffices to find a positive integer n such that it is independent on the choices of the maximal ideals m. However, we have the following useful corollary.

Corollary 4.4. *Let R be an excellent domain of dimension $d > 0$ and x be an element of R° such that R_x is a CM ring. If T is a finite set of maximal ideals of R, then there exists a positive integer n such that for any $m \in T$, $x^n \mathrm{H}_m^i(R) = 0$ for $i < \mathrm{ht}(m)$.*

To simplify the proof of the main result of this section, we need the following lemma, which enable us to obtain the annihilators of local cohomology modules.

Lemma 4.5. *Let (R, m) be a noetherian local ring of dimension d. Let $x_1, x_2, \dots, x_r$ be a part of system of parameters in m and x an element in m. Suppose that*

(i) $R/(x_1^{n_1}, x_2^{n_2}, \dots, x_r^{n_r})$ *are CM*;
(ii) *For* $1 \leqslant i \leqslant r$, $x((x_1^{n_1}, x_2^{n_2}, \dots, x_{i-1}^{n_{i-1}}) : x_i^{n_i}) \subseteq (x_1^{n_1}, x_2^{n_2}, \dots, x_{i-1}^{n_{i-1}})$

hold for all positive integers $n_1, n_2, \dots, n_r$. Then $x^r H_m^i(R) = 0$ for $i < d$.

Proof. We will use induction on j $(0 \leqslant j \leqslant r)$ to assert that for any positive integers $n_1, n_2, \dots, n_j$

$$x^{r-j}\big(\mathrm{H}_m^i\big(R/\big(x_1^{n_1}, x_2^{n_2}, \dots, x_j^{n_j}\big)\big)\big) = 0$$

for $i < d - j$, and then the lemma follows if we set $j = 0$.

By the assumption, for arbitrary fixed integers $n_1, n_2, \dots, n_r$, $R/(x_1^{n_1}, x_2^{n_2}, \dots, x_r^{n_r})$ is *CM*, so the conclusion is trivial in this case. Suppose that we have proved the conclusion for $t + 1 \leqslant j \leqslant r$. For a fixed i with $i < d - t$, let z be an arbitrary element in $\mathrm{H}_m^i(R/(x_1^{n_1}, x_2^{n_2}, \dots, x_t^{n_t}))$. Choose a positive integer n_{t+1} such that $x_{t+1}^{n_{t+1}} z = 0$. Put

(1) $R_t = R/(x_1^{n_1}, x_2^{n_2}, \dots, x_t^{n_t})$;
(2) $R_{t+1} = R/(x_1^{n_1}, x_2^{n_2}, \dots, x_{t+1}^{n_{t+1}})$;
(3) $U_t = ((x_1^{n_1}, x_2^{n_2}, \dots, x_t^{n_t}) : x_{t+1}^{n_{t+1}})/(x_1^{n_1}, x_2^{n_2}, \dots, x_t^{n_t})$;
(4) $N_t = x_{t+1}^{n_{t+1}}(R/(x_1^{n_1}, x_2^{n_2}, \dots, x_t^{n_t}))$.

Let us consider the short exact sequences

$$0 \to N_t \to R_t \to R_{t+1} \to 0,$$
$$0 \to U_t \to R_t \to N_t \to 0,$$

we have the following long exact sequences of local cohomology

$$\cdots \to \mathrm{H}_m^{i-1}(R_{t+1}) \to \mathrm{H}_m^i(N_t) \xrightarrow{\phi_i} \mathrm{H}_m^i(R_t) \to \cdots,$$
$$\cdots \to \mathrm{H}_m^i(U_t) \to \mathrm{H}_m^i(R_t) \xrightarrow{\psi_i} \mathrm{H}_m^i(N_t) \to \cdots.$$

It is easy to see that the composition $\phi_i \psi_i$ of ψ_i and ϕ_i is the morphism

$$\mathrm{H}_m^i(R_t) \xrightarrow{x_{t+1}^{n_{t+1}}} \mathrm{H}_m^i(R_t)$$

and thus $\phi_i(\psi_i(z)) = 0$. By the induction hypothesis, $x^{r-t-1}(\mathrm{H}_m^{i-1}(R_{t+1})) = 0$, so from the first long exact sequence above, we conclude that $(\psi_i(x^{r-t-1}z)) = 0$. The condition (ii) implies that $xU_t = 0$, so $x\mathrm{H}_m^i(U_t) = 0$. Hence from the second long exact sequence above, we have

$x^{r-t}z = 0$. By the choice of z, we have proved $x^{r-t}(\mathrm{H}^i_m(R_t)) = 0$. This ends the inductive proof of the lemma. □

In the rest of the paper, we will make use of Proposition 4.2, Corollary 4.4 and Lemma 4.5 to prove the following main result of this section.

Theorem 4.6. *Let R be a locally equidimensional excellent ring of dimension d. If $d \leqslant 5$, then R has a uniform local cohomological annihilator.*

Proof. By Theorem 3.2, we may assume that R is an excellent domain. Moreover, let S be the integral closure of R in the field of fractions of R. Since R is excellent, it follows that S is a finitely generated R-module. We first conclude that if S has a uniform local cohomological annihilator y, then R also has a uniform local cohomological annihilator.

In fact, as y is integral over R, we have

$$y^s + a_1 y^{s-1} + \cdots + a_s = 0$$

for some elements $a_1, \dots, a_s$ contained in R with $a_s \neq 0$. So it is clear a_s is a uniform local cohomological annihilator of S. Consider the following natural short exact sequence

$$0 \to R \to S \to M \to 0, \tag{4.1}$$

where M is a finitely generated R-module. It is easy to find a nonzero element x of R such that $xM = 0$. Now, for an arbitrary maximal m of R, all the minimal prime ideals $Q_1, Q_2, \dots, Q_t$ of mS are maximal. Thus by the Mayer–Vietoris sequence of local cohomology, we conclude that

$$\mathrm{H}^i_m(S) \simeq \mathrm{H}^i_{Q_1}(S) \oplus \cdots \oplus \mathrm{H}^i_{Q_t}(S)$$

and consequently $a_s \mathrm{H}^i_m(S) = 0$ for $i < \mathrm{ht}(m)$. From the long exact sequence of local cohomology induced from (4.1), we have $a_s x \mathrm{H}^i_m(R) = 0$ for $i < \mathrm{ht}(m)$. Thus $a_s x$ is a uniform local cohomological annihilator of R. So in the following proof we assume R is a normal domain.

If $d \leqslant 2$, then R is *CM*, and there is nothing to prove. So we assume $d > 2$. Set $V = \{P \in \mathrm{Spec}(R) \mid R_P \text{ is not a } CM \text{ ring}\}$. As R is an excellent ring, the *CM* locus of R is open in $\mathrm{Spec}(R)$, so there exists an ideal I of R such that $V = V(I)$. Clearly, $\mathrm{ht}(I) \geqslant 3$. Choose elements x_1, x_2, x_3 contained in I such that $\mathrm{ht}((x_i, x_j)) = 2$ for $i \neq j$, and $\mathrm{ht}((x_1, x_2, x_3)) = 3$.

Claim 1. *The union T_1 of the sets of associated prime ideals $\mathrm{Ass}_R(R/(x_1^{n_1}, x_2^{n_2}))$ is a finite set, where the union is taken over all positive integers n_1, n_2.*

Proof. It is well known the union T_0 of the sets of associated prime ideals $\mathrm{Ass}_R(R/(x_1, x_2^{n_2}))$ is a finite set, where the union is taken over all positive integers n_2. We assert that $T_0 = T_1$. By induction on n_1, this follows easily from the short exact sequences

$$0 \to R/\big(x_1, x_2^{n_2}\big) \to R/\big(x_1^{n_1}, x_2^{n_2}\big) \to R/\big(x_1^{n_1-1}, x_2^{n_2}\big) \to 0$$

for $n_1 > 1$. This proves the claim. □

Let T_2 denote the union of T_1 and the set of minimal prime ideals of (x_1, x_2, x_3). Clearly, T_2 is a finite set. Since R_{x_3} is *CM*, by [HH2, Lemma 3.2], we can choose a positive number n such that, for all positive integers n_1, n_2, the following holds

$$\big(\big(x_1^{n_1}, x_2^{n_2}\big) : x_3^n\big)_P = \big(\big(x_1^{n_1}, x_2^{n_2}\big) : x_3^{n+1}\big)_P \tag{4.2}$$

for every prime ideal P lies in the set T_2.

For any prime ideal P with $(x_1, x_2, x_3) \subseteq P$ and $P \notin T_2$, it is clear $\mathrm{ht}(P) \geqslant 4$ and $\mathrm{depth}(R_P/(x_1^{n_1}, x_2^{n_2})_P) \geqslant 1$. Localizing at P if necessary, we may assume P is a maximal ideal. Giving an element $c \in R$ satisfying $Pc \subseteq ((x_1^{n_1}, x_2^{n_2}) : x_3^n)$, it follows $Px_3^n c \subseteq (x_1^{n_1}, x_2^{n_2})$. By the choice of P, we have $x_3^n c \in (x_1^{n_1}, x_2^{n_2})$, and so we conclude that P is not an associated prime of $R/((x_1^{n_1}, x_2^{n_2}) : x_3^n)$. Therefore, the associated prime ideals $R/((x_1^{n_1}, x_2^{n_2}) : x_3^n)$ are contained in T_2. Thus by (4.2), we conclude that

$$\big(\big(x_1^{n_1}, x_2^{n_2}\big) : x_3^n\big) = \big(\big(x_1^{n_1}, x_2^{n_2}\big) : x_3^{n+1}\big)$$

hold for all positive integers n_1, n_2.

Similarly, enlarging n if necessary, one can prove

$$\big(\big(x_2^{n_2}, x_3^{n_3}\big) : x_1^n\big) = \big(\big(x_2^{n_2}, x_3^{n_3}\big) : x_1^{n+1}\big),$$
$$\big(\big(x_1^{n_1}, x_3^{n_3}\big) : x_2^n\big) = \big(\big(x_1^{n_1}, x_3^{n_3}\big) : x_2^{n+1}\big).$$

Replacing x_1, x_2, x_3 by x_1^n, x_2^n, x_3^n respectively, we have proved

$$\begin{aligned} \big(\big(x_1^{n_1}, x_2^{n_2}\big) : x_3\big) &= \big(\big(x_1^{n_1}, x_2^{n_2}\big) : x_3^2\big), \\ \big(\big(x_2^{n_2}, x_3^{n_3}\big) : x_1\big) &= \big(\big(x_2^{n_2}, x_3^{n_3}\big) : x_1^2\big), \\ \big(\big(x_1^{n_1}, x_3^{n_3}\big) : x_2\big) &= \big(\big(x_1^{n_1}, x_3^{n_3}\big) : x_2^2\big) \end{aligned} \tag{4.3}$$

hold for all positive integers n_1, n_2, n_3.

If $\dim(R/(x_1, x_2, x_3)) = 0$, then there are only finite number of maximal ideals m such that R_m may not be a *CM* ring. So by Corollary 4.4, a power of x_1 is a uniform local cohomological annihilator of R. In particular, the conclusion of the theorem holds for $d = 3$.

Now, in the following we assume $\dim(R/(x_1, x_2, x_3)) > 0$. Since R is excellent, we can choose an element x_4 such that x_4 is not contained in any minimal prime ideal of (x_1, x_2, x_3) and $(R/(x_1^2, x_2^2, x_3^2))_P$ is *CM* for all prime ideals P with $x_4 \notin P$. For such a prime ideal P, we conclude by Proposition 4.2 that $(R/(x_1^2, x_2^2, x_3^{n_3}))_P$ are all *CM* rings for all positive integers n_3. By (4.3) and by Proposition 4.2 again, we conclude that $(R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}))_P$ are all *CM* rings for all positive integers n_1, n_2, n_3.

Claim 2. *Let m be a maximal ideal of R such that m does not contain the ideal (x_1, x_2, x_3, x_4). Then for every x_j $(1 \leqslant j \leqslant 3)$, $x_j^3 \mathrm{H}_m^i(R) = 0$ for $i < \mathrm{ht}(m)$.*

Proof. Let m be an arbitrary maximal ideal with $x_4 \notin m$. If one of x_1, x_2, x_3 does not lie in m, then R_m is *CM*, and the conclusion is trivial if one of them does not lie in m. So we may assume

$m \supseteq (x_1, x_2, x_3)$. Clearly, we may also assume that R is a d-dimensional local ring with the unique maximal ideal m.

Note that $R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3})$ is *CM* of dimension $d-3$ for any positive integers n_1, n_2, n_3. Moreover, x_1, x_2 is a regular sequence in R, so together with (4.3), we have for $1 \leqslant i \leqslant 3$, $x_j((x_1^{n_1}, \dots, x_{i-1}^{n_{i-1}}) : x_i^{n_i}) \subseteq (x_1^{n_1}, \dots, x_{i-1}^{n_{i-1}})$ for each j with $1 \leqslant j \leqslant 3$. Hence the conclusion of the claim follows immediately from Lemma 4.5, and this proves the claim. $\square$

By Claim 2, we have proved that if a maximal ideal m does not contain the ideal (x_1, x_2, x_3, x_4), then for each j $(1 \leqslant j \leqslant 3)$, $x_1^3 H_m^i(R) = 0$ for $i < \mathrm{ht}(m)$. Now, suppose that $\dim(R/(x_1, x_2, x_3, x_4)) = 0$. There are only a finite number of maximal ideals m with $(x_1, x_2, x_3, x_4) \subseteq m$. So by Claim 2 and Corollary 4.4, a power of x_j $(1 \leqslant j \leqslant 3)$ is a uniform local cohomological annihilator of R. In particular, the conclusion of the theorem holds for $d = 4$. In the following proof, we assume $\dim(R/(x_1, x_2, x_3, x_4)) > 0$.

Claim 3. *Let T_3 be the set of all associated prime ideals of $R/(x_1^{i_1}, x_2^{i_2}, x_3^{i_3})$ for all positive integers i_1, i_2, i_3 satisfying $i_1 + i_2 + i_3 \leqslant 6$. Then for any positive integers n_1, n_2, n_3, every associated prime ideal of $R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3})$ lies in T_3.*

Proof. We prove the conclusion by induction on $n = n_1 + n_2 + n_3$. Clearly, if $n \leqslant 6$, the conclusion holds by the assumption. In the following we assume $n > 6$. Suppose we have proved the conclusion for $n_1 + n_2 + n_3 < n$. It is easy to see there exists one of n_1, n_2, n_3, say n_3, such that $n_3 \geqslant 3$.

Suppose that P is an associated prime ideal of $R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3})$ and $P \notin T_3$. Localizing at P if necessary, we may assume that R is a local ring with the unique maximal ideal P. Let $c \notin (x_1^{n_1}, x_2^{n_2}, x_3^{n_3})$ be an element satisfying

$$Pc \subseteq \left(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}\right).$$

Then we can express $c = x_1^{n_1} c_1 + x_2^{n_2} c_2 + x_3^{n_3 - 1} c_3$ by the induction hypothesis. For an arbitrary element $z \in P$, it implies that there exists $c_4 \in R$ such that

$$x_3^{n_3 - 1}(z c_3 - x_3 c_4) \in \left(x_1^{n_1}, x_2^{n_2}\right).$$

By (4.3), we have $z x_3 c_3 \in (x_1^{n_1}, x_2^{n_2}, x_3^2)$. Hence $P x_3 c_3 \subseteq (x_1^{n_1}, x_2^{n_2}, x_3^2)$. Since $n_1 + n_2 + 2 < n$, we conclude $x_3 c_3 \in (x_1^{n_1}, x_2^{n_2}, x_3^2)$ by the induction hypothesis again. Consequently $c \in (x_1^{n_1}, x_2^{n_2}, x_3^{n_3})$, and this is a contradiction. Therefore, $P \in T_3$, and the proof of the claim is complete. $\square$

Let T_4 be the set of the union of T_3 and the set of all minimal prime ideal (x_1, x_2, x_3, x_4). It is clear that T_4 is a finite set. By [HH2, Lemma 3.2], we can choose a positive integer n such that for every prime ideal $P \in T_4$

$$\left(\left(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}\right) : x_4^{n_4}\right)_P \subseteq \left(\left(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}\right) : x_j^{n}\right)_P$$

for all positive integers n_1, n_2, n_3, n_4 and $j \leqslant 3$. Replacing x_j by x_j^n, we have

$$\left(\left(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}\right) : x_4^{n_4}\right)_P \subseteq \left(\left(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}\right) : x_j\right)_P \tag{4.4}$$

for all positive integers n_1, n_2, n_3, n_4 and $j \leqslant 3$.

For $P \notin T_4$, if $Pc \subseteq ((x_1^{n_1}, x_2^{n_2}, x_3^{n_3}) : x_j)$ for some element $c \in R$, then $Px_jc \in (x_1^{n_1}, x_2^{n_2}, x_3^{n_3})$. Thus $x_jc \in (x_1^{n_1}, x_2^{n_2}, x_3^{n_3})$ by the choice of P. This shows that, for all positive integers n_1, n_2, n_3, the associated primes of $R/((x_1^{n_1}, x_2^{n_2}, x_3^{n_3}) : x_j)$ lie in T_4. From this fact and (4.4), one can conclude easily that

$$\left(\left(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}\right) : x_4^{n_4}\right) \subseteq \left(\left(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}\right) : x_j\right) \tag{4.5}$$

hold for all positive integers n_1, n_2, n_3, n_4 and $j \leqslant 3$.

Claim 4. *For every permutation i_1, i_2, i_3 of $1, 2, 3$, we have*

$$\left(\left(x_{i_1}^{n_1}, x_{i_2}^{n_2}, x_4^{n_4}\right) : x_{i_3}^2\right) = \left(\left(x_{i_1}^{n_1}, x_{i_2}^{n_2}, x_4^{n_4}\right) : x_{i_3}^3\right)$$

hold for all positive integers n_1, n_2, n_4.

Proof. We only prove the following case:

$$\left(\left(x_1^{n_1}, x_2^{n_2}, x_4^{n_4}\right) : x_3^2\right) = \left(\left(x_1^{n_1}, x_2^{n_2}, x_4^{n_4}\right) : x_3^3\right).$$

In fact, for any element $c \in ((x_1^{n_1}, x_2^{n_2}, x_4^{n_4}) : x_3^3)$, we may express

$$x_3^3 c = x_1^{n_1} c_1 + x_2^{n_2} c_2 + x_4^{n_4} c_4$$

for some elements $c_1, c_2, c_4 \in R$. So by (4.5), there exists $c_3 \in R$ such that $x_3^3(x_3c - x_4^{n_4}c_3) \in (x_1^{n_1}, x_2^{n_2})$. By (4.3), we have $x_3^2 c \in (x_1^{n_1}, x_2^{n_2}, x_4^{n_4})$, and this proves the claim. □

Replacing x_1, x_2, x_3 by x_1^2, x_2^2, x_3^2 if necessary, we may assume

$$\left(\left(x_{i_1}^{n_1}, x_{i_2}^{n_2}, x_4^{n_4}\right) : x_{i_3}\right) = \left(\left(x_{i_1}^{n_1}, x_{i_2}^{n_2}, x_4^{n_4}\right) : x_{i_3}^2\right) \tag{4.6}$$

for every permutation i_1, i_2, i_3 of $1, 2, 3$ and for all positive integers n_1, n_2, n_4.

Moreover, we can replace x_4 by a power of x_4 if necessary, and assume that $((x_1^2, x_2^2, x_3^2) : x_4) = ((x_1^2, x_2^2, x_3^2) : x_4^2)$. Since R is an excellent ring, we can choose an element x_5 which is not contained in any minimal prime ideal of (x_1, x_2, x_3, x_4) such that $(R/(x_1^2, x_2^2, x_3^2, x_4^2))_P$ is a *CM* ring for every prime ideal P with $x_5 \notin P$. By the choice of x_4 and Proposition 4.2, we conclude that $(R/(x_1^2, x_2^2, x_3^2, x_4^{n_4}))_P$ is a *CM* ring for every prime ideal P with $(x_1, x_2, x_3, x_4) \subseteq P$, $x_5 \notin P$ and all positive integers n_4. It follows from (4.6) and Proposition 4.2 that $(R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, x_4^{n_4}))_P$ are *CM* local rings for such prime ideals P and all positive integers n_1, n_2, n_3, n_4.

Claim 5. *Let m be a maximal ideal of R such that m does not contain the ideal $(x_1, x_2, x_3, x_4, x_5)$. Then for every x_j $(1 \leqslant j \leqslant 3)$, $x_j^5 \mathrm{H}_m^i(R) = 0$ for $i < \mathrm{ht}(m)$.*

Proof. Note that if m does not contain the ideal (x_1, x_2, x_3, x_4), then the conclusion of the claim follows from Claim 2. So we may assume that $(x_1, x_2, x_3, x_4) \subseteq m$ and $x_5 \notin m$. Replacing R

by R_m, we assume that R is a local ring with the maximal ideal m. Observe that x_1, x_2 is a regular sequence, (4.3) and (4.5). We have for $1 \leqslant i \leqslant 4$

$$x_j\big(\big(x_1^{n_1}, x_2^{n_2}, \ldots, x_{i-1}^{n_{i-1}}\big) : x_i^{n_i}\big) \subseteq \big(x_1^{n_1}, x_2^{n_2}, \ldots, x_{i-1}^{n_{i-1}}\big)$$

for each j with $1 \leqslant j \leqslant 3$. Moreover, by the choice of m, $R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, x_4^{n_4})$ are *CM* local rings for all positive integers n_1, n_2, n_3, n_4. So the conclusion of the claim follows from Lemma 4.5, and this ends of the proof of the claim. □

Now, if $\dim(R/(x_1, x_2, x_3, x_4, x_5)) = 0$, then there are only a finite number of maximal ideals m with $(x_1, x_2, x_3, x_4, x_5) \subseteq m$. So by Claim 5, and Corollary 4.4, a power of x_j $(1 \leqslant j \leqslant 3)$ is a uniform local cohomological annihilator of R. In particular, we have proved the theorem in the case $d = 5$. □

Before the end of the paper, we give a remark on the technique used in this section.

Remark. In the proof of Theorem 4.6, we depend heavily on one of Goto's results (Proposition 4.2). The condition $(0 : x) = (0 : x^2)$ in Proposition 4.2 is very restricted if one considers a lot of local rings at the same time. We explain this more explicitly by means of the proof of Theorem 4.6. Let R, x_1, x_2, x_3, x_4 and x_5 be as chosen as in the proof of Theorem 4.6. Although we can find an element x_6 such that for all maximal ideals m with $x_6 \notin m$, $R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, x_4^{n_4}, x_5)$ are *CM* rings for all positive integers n_1, n_2, n_3, n_4, it is very difficult to check that

$$\big(\big(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, x_4^{n_4}\big) : x_5\big) = \big(\big(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, x_4^{n_4}\big) : x_5^2\big)$$

hold for all positive integers n_1, n_2, n_3, n_4. So one cannot use Proposition 4.2 to conclude that $R/(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, x_4^{n_4}, x_5^{n_5})$ are *CM* rings for all positive integers n_1, n_2, n_3, n_4, n_5. Thus the method of this paper cannot be used to solve the remaining case of Conjecture 4.1. However, with a little more effort, we can prove that, after replacing x_1, x_2, x_3, x_4, x_5 by suitable powers of them, for each j $(1 \leqslant j \leqslant 3)$

$$x_j\big(\big(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, x_4^{n_4}\big) : x_5^{n_5}\big) \subseteq \big(x_1^{n_1}, x_2^{n_2}, x_3^{n_3}, x_4^{n_4}\big)$$

hold for all positive integers n_1, n_2, n_3, n_4, n_5. Due to this fact, one can prove easily that for any excellent domain R and an element of x in R such that R_x is *CM*, there exists a positive integer n, $x^n \mathrm{H}_m^i(R) = 0$ for every maximal ideal m and $i < \min(5, \mathrm{ht}(m)) - 1$.

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